Stability and multiple bifurcations of a damped harmonic oscillator with delayed feedback near zero eigenvalue singularity

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We investigate the dynamics of a damped harmonic oscillator with delayed feedback near zero eigenvalue singularity. We perform a linearized stability analysis and multiple bifurcations of the zero solution of the system near zero eigenvalue singularity. Taking the time delay as the bifurcation parameter, the presence of steady-state bifurcation, Bogdanov–Takens bifurcation, triple zero, and Hopf-zero singularities is demonstrated. In the case when the system has a simple zero eigenvalue, center manifold reduction and normal form theory are used to investigate the stability and the types of steady-state bifurcation. The stability of the zero solution of the system near the simple zero eigenvalue singularity is completely solved. © *2008 American Institute of Physics*.
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Time delay is commonly encountered in biological, mechanical, and electronic systems. For the delayed systems, a natural question is how the time delay affects the stability and induces possible oscillations and complex dynamics. It is well know that the dynamics of delayed systems near zero eigenvalue singularity is often complex and hard to investigate. Here we study the dynamics of differential delay equations that arise in the delayed feedback control of mechanical systems, especially its dynamics near zero eigenvalue singularity. Taking the time delay as the bifurcation parameter, we not only investigate the influence of the time delay on the stability but also demonstrate the presence of steady-state bifurcation, Bogdanov–Takens bifurcation, triple zero, and Hopf-zero singularities. In the situation that the zero is a simple eigenvalue, the normal forms of the reduced equations are obtained by the center manifold theory and normal form method for functional differential equation, and hence the stability of the fixed point is determined, and transcritical and pitchfork bifurcations are found.

I. INTRODUCTION

In classical mechanics, a harmonic oscillator is a system which, when displaced from its equilibrium position, experiences a restoring force f. The harmonic oscillator has been playing a very important role in the study of nonlinear dynamics. Early studies have shown that the harmonic oscillator is truly able to exhibit remarkable complex dynamical behavior. If a frictional force (damping) proportional to the velocity is also present, the harmonic oscillator is described as a damped oscillator. Time delays are intrinsic and important features of many physical and biological control systems. The time delays most commonly occur as a consequence of finite conduction and production times. So, the time delay has been introduced to the damped harmonic oscillator.[1–4] Using Newton's Second Law of motion, the dynamics of the damped harmonic oscillator with a delayed restoring force is described by the second delay differential equation

$$\ddot{x}(t) + b\dot{x}(t) + ax(t) = f[x(t-\tau)], \tag{1.1}$$

where a, b are constants, τ is the time delay, $x(t)$, $x(t-\tau)$ are the displacement at times t, $t-\tau$, respectively, and the function, f, describes the feedback. Equation (1.1) also arises in a variety of mechanical, or neuromechanical, system in which inertia plays an important role.[5–11] The linear stability and Hopf bifurcation of Eq. (1.1) have been investigated by many authors (see Refs. 1, 4, and 10–12, and references therein). The complex dynamics of Eq. (1.1), including chaos and two-tori, have been discussed by Boe and Chang[10,11] for the case when f is a nonmonotone function. Limit cycles, the nature of the bifurcations of the two-tori, and multistability have been investigated by Campbell *et al.*[1–3] when f is simple monotone negative feedback. However, most of the works cited above have been done on systems of the form (1.1) under the assumption that the system does not have a zero eigenvalue. Thus, the natural question is what happens when the system has a zero eigenvalue. The main purpose of this paper is to study the stability and bifurcations of system (1.1) when it has a zero eigenvalue, especially a simple zero eigenvalue.

This paper is organized as follows: The local stability analysis and multiple bifurcations of system (1.1) with a zero eigenvalue is investigated in Sec. II. The normal form on the center manifold and the dynamics near the simple zero eigenvalue are discussed in Sec. III. In Sec. IV, numerical simulations are performed to illustrate the results. The paper ends with conclusions, in Sec. V.

II. LINEAR STABILITY ANALYSIS AND MULTIPLE BIFURCATIONS

Equation (1.1) can be written as a first order delay functional differential system of the form

$$\begin{aligned}\dot{x}(t) &= y(t),\\ \dot{y}(t) &= -ax(t) - by(t) + f[x(t-\tau)].\end{aligned} \tag{2.1}$$

Setting $d=f'(0)$, then the linearized equation at the origin is

$$\dot{x}(t) = Mx(t) + Nx(t-\tau), \tag{2.2}$$

where

$$M = \begin{pmatrix} 0 & 1 \\ -a & -b \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix}. \tag{2.3}$$

Thus, the characteristic equation of Eq. (2.2) is given by

$$\Delta(\lambda,\tau) = \lambda^2 + b\lambda + a - de^{-\lambda\tau} = 0. \tag{2.4}$$

Lemma 2.1: Assuming that $d=a$ and $b<0$, then we have the following:

- *(i) when $\tau \neq -b/a$, $\lambda=0$ is a simple zero root of the characteristic Eq. (2.4);*
- *(ii) when $\tau=-b/a$ and $a\neq b^2/2$, $\lambda=0$ is a double zero root of the characteristic Eq. (2.4);*
- *(iii) when $\tau=-b/a$ and $a=b^2/2$, $\lambda=0$ is a triple zero root of the characteristic Eq. (2.4).*

Proof: The presence of zero roots follows from the fact that $\Delta(0,\tau)=0$ when $d=a$. Substituting $d=a$ into $\Delta(\lambda,\tau)$ and taking the partial derivative with respective to λ yields

$$\frac{\partial\Delta(\lambda,\tau)}{\partial\lambda} = 2\lambda + b + \tau a e^{-\lambda\tau}.$$

Clearly, $\partial\Delta(0,\tau)/\partial\lambda=0$ if and only if $\tau=-b/a$. So, the conclusion (i) follows.

In addition, we also have

$$\frac{\partial^2\Delta(\lambda,\tau)}{\partial\lambda^2} = 2 - \tau^2 a e^{-\lambda\tau}$$

and

$$\frac{\partial^3\Delta(\lambda,\tau)}{\partial\lambda^3} = \tau^3 a e^{-\lambda\tau},$$

which implies that $\partial^2\Delta(0,\tau)/\partial\lambda^2=0$ if and only if $\tau=-b/a$ and $a=b^2/2$, but $\partial^3\Delta(0,\tau)/\partial\lambda^3\neq 0$. This completes the proofs of (ii) and (iii). □

Lemma 2.2: (I) *Assuming that $d=a>b^2/2$, $b<0$ and $\tau_0=1/\sqrt{2a-b^2}\arccos((b^2-a)/a)$, we have the following*:

- *(i) when $0\leq\tau<-b/a$, Eq. (2.4) has exactly one positive real root;*
- *(ii) when $\tau=-b/a$, all roots of Eq. (2.4), except for the double zero roots, have negative real parts;*
- *(iii) when $-b/a<\tau<\tau_0$, all roots of Eq. (2.4), except for the single zero root, have negative real parts;*
- *(iv) when $\tau=\tau_0$, all roots of Eq. (2.4), except for the single zero root and a pair of purely imaginary roots $\pm i\sqrt{2a-b^2}$, have negative real parts;*
- *(v) when $\tau>\tau_0$, Eq. (2.4) has at least a pair of roots with positive real parts.*

(II) *If $d=a=b^2/2$, $b<0$, then we have the following*:

- *(i) when $0\leq\tau<-b/a$, Eq. (2.4) has exactly one positive real root;*
- *(ii) when $\tau=-b/a$ all roots of Eq. (2.4), except for the triple zero root, have negative real parts;*
- *(iii) when $\tau>-\frac{b}{a}$, Eq. (2.4) has at least one root with positive real parts.*

(III) *If $0<d=a<b^2/2$ and $b<0$, then Eq. (2.4) has at least one root with positive real parts for all $\tau\geq 0$.*

Proof: Clearly, when $\tau=0$, Eq. (2.4) with $d=a$ has a zero root $\lambda_1=0$ and a positive real root $\lambda_2=-b$. Suppose that $i\omega(\omega>0)$ is a root of Eq. (2.4) with $d=a$, then ω satisfies

$$a-\omega^2 = a\cos(\omega\tau), \quad b\omega = -a\sin(\omega\tau), \tag{2.5}$$

which yields

$$\omega^2 = 2a - b^2. \tag{2.6}$$

So, if $a\leq\frac{1}{2}b^2$ Eq. (2.6) has no positive real root, and if $a>\frac{1}{2}b^2$ Eq. (2.6) has one positive real root

$$\omega_+ = \sqrt{2a-b^2}.$$

For $a>\frac{1}{2}b^2$, set

$$\tau_k = \frac{1}{\omega_+}\left[\arccos\left(\frac{b^2-a}{a}\right) + 2k\pi\right], \quad k=0,1.$$

Equation (2.4) with $d=a$ has a pair of purely imaginary roots $\pm i\omega_+$ when $\tau=\tau_k$. From Eq. (2.5), we have

$$\sin\left[\frac{a\tau_0}{-b}\left(\frac{-b}{a}\omega_+\right)\right] = \frac{-b}{a}\omega_+,$$

which implies that

$$\tau_0 > \frac{-b}{a}, \quad \text{for } a > \frac{b^2}{2}, \tag{2.7}$$

and

$$\lim_{a\to(b^2/2)^+}\tau_0 = \lim_{a\to(b^2/2)^+}\frac{\arccos\left(\frac{b^2-a}{a}\right)}{\sqrt{2a-b^2}} = \frac{-b}{a}. \tag{2.8}$$

Denoted by

$$\lambda(\tau) = \eta(\tau) + i\omega(\tau)$$

the root of Eq. (2.4) with $d=a$ satisfy

$$\eta(\tau_k)=0, \quad \omega(\tau_k)=\omega_+.$$

Differentiating Eq. (2.4) with respect to τ gives

$$\frac{d\lambda}{d\tau} = \frac{-a\lambda e^{-\lambda\tau}}{2\lambda + b + \tau a e^{-\lambda\tau}} \tag{2.9}$$

which, together with Eq. (2.4), leads to

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{2\lambda + b}{-\lambda(\lambda^2 + b\lambda + a)} - \frac{\tau}{\lambda},$$

and then

$$\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_k} = \frac{\omega_+}{b^2\omega_+^2 + (a-b^2)^2} > 0.$$

So, we have the following transversality condition:

$$\mathrm{sgn}\left[\frac{d}{d\tau}\mathrm{Re}\,\lambda(\tau_k)\right] = \mathrm{sgn}\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_k}\right] > 0. \qquad (2.10)$$

Conclusion (v) of (I) immediately follows from the above inequality.

From Eq. (2.9), we have

$$\left.\frac{d\lambda}{d\tau}\right|_{\lambda=0} = \left.\frac{-a\lambda e^{-\lambda\tau}}{2\lambda + b + \tau a e^{-\lambda\tau}}\right|_{\lambda=0} = 0, \quad \text{for } \tau \neq \frac{-b}{a}, \qquad (2.11)$$

which, together with Rouché theorem and the fact that Eq. (2.6) has no positive real root for $a \leqslant \frac{1}{2}b^2$, implies conclusion (III) is true. From Eq. (2.9), we also have

$$\begin{aligned}\lim_{\lambda\to 0}\frac{d\tau}{d\lambda} &= \lim_{\lambda\to 0}\frac{2\lambda + b + \tau a e^{-\lambda\tau}}{-a\lambda e^{-\lambda\tau}} \\ &= \lim_{\lambda\to 0}\frac{2 - \tau^2 a e^{-\lambda\tau}}{(-a + \tau a\lambda)e^{-\lambda\tau}} \\ &= \frac{\tau^2 a - 2}{a}\begin{cases} <0, & \text{for } \tau = \dfrac{-b}{a}, \quad a > \dfrac{b^2}{2}, \\ >0, & \text{for } \tau = \dfrac{-b}{a}, \quad a < \dfrac{b^2}{2}. \end{cases}\end{aligned} \qquad (2.12)$$

The combination of Eqs. (2.7), (2.8), (2.11), and (2.12) and the Rouché theorem completes the proofs of (i)–(iv) of (I) and (i) of (II).

When (a_1, b_1) lies on the curve $a=b^2/2$, suppose for a contradiction that Eq. (2.4) with $\tau=(-b_1)/a_1$ has a root with a positive real part, say $\eta_0+i\omega_0$ with $\eta_0>0$. Let $\lambda(a) = \eta(a)+i\omega(a)$ be the root of Eq. (2.4) satisfying $\eta(a_1)=\eta_0 > 0$ and $\omega(a_1)=\omega_0$. There exists a sufficiently small $\delta>0$ such that Eq. (2.4) with $a=a_1+\delta$, $b=b_1$ has a root with positive real part at $\tau=-b_1/(a_1+\delta)$, which contradicts (ii) of (II). This completes the proof of (ii) of (II). In addition, we also have

$$\lim_{\lambda\to 0}\frac{d\tau}{d\lambda} = \frac{\tau^2 a - 2}{a}\begin{cases} =0, & \text{for } \tau = \dfrac{-b}{a}, \quad a = \dfrac{b^2}{2}, \\ >0, & \text{for } \tau > \dfrac{-b}{a}, \quad a = \dfrac{b^2}{2}. \end{cases}$$

So (iii) of (II) is also true. The proof is complete.

By Lemmas 2.1, 2.2, and the theory of the functional differential equations, we have the following two theorems.

Theorem 2.1: (I) *Assuming that* $d=a>b^2/2$, $b<0$, and $\tau_k=1/\sqrt{2a-b^2}[\arccos((b^2-a)/a)+2k\pi]$, $k=0,1,2,\ldots,$ *we have the following*:

(i) *when* $\tau\neq -b/a$ *and* $\tau\neq\tau_k$, *Eq. (1.1) undergoes a codimension-one steady state bifurcation at the zero steady state;*

(ii) *when* $\tau=-b/a$, *Eq. (1.1) undergoes a Bogdanov-Takens bifurcation at the zero steady state;*

(iii) *when* $\tau=\tau_k$, *Eq. (1.1) undergoes a steady-state/Hopf interaction at the zero steady state;*

(II) *If* $d=a=b^2/2$, $b<0$, *then we have the following:*

(i) *when* $\tau\neq -b/a$, *Eq. (1.1) undergoes a codimension-one steady state bifurcation at the zero steady state;*

(ii) *when* $\tau=-b/a$, *the zero steady state has a triple zero eigenvalue singularity;*

(III) *If* $0<d=a<b^2/2$ *and* $b<0$, *then Eq.* (1.1) *undergoes a codimension-one steady state bifurcation at the zero steady state for all* $\tau\neq -b/a$ *and undergoes a Bogdanov-Takens bifurcation for* $\tau=-b/a$.

Theorem 2.2:

(i) *If* $d=a>b^2/2$, $b<0$ *and* $\tau_0=1/\sqrt{2a-b^2}\arccos((b^2-a)/a)$, *then the zero steady state of Eq. (1.1) is unstable when* $0\leqslant\tau<-b/a$ *or* $\tau>\tau_0$;

(ii) *If* $d=a=b^2/2$ *and* $b<0$, *then the zero steady state of Eq. (1.1) is unstable when* $\tau\neq -b/a$;

(iii) *If* $0<d=a<b^2/2$ *and* $b<0$, *then the zero steady state of Eq. (1.1) is unstable for all* $\tau\geqslant 0$.

Remark 2.1: All bifurcations in the considered system are nongeneric, since there is always the trivial equilibrium. Therefore, one cannot apply results on generic fold, Bogdanov–Takens or fold-Hopf bifurcations. As you will see in the following section, in the simplest case of one critical eigenvalue zero, we found only the transcritical and pitchfork bifurcations, but not a fold singularity.

Remark 2.2: From Lemma 2.2, we know that if either $d=a>b^2/2$, $b<0$ *and* $-b/a\leqslant\tau\leqslant\tau_0$, *or* $d=a>b^2/2$, $b<0$, *and* $\tau=-b/a$, *all roots of Eq. (2.4), except for roots with zero real parts, have negative real parts. So, in these cases, it is not sufficient to determine the stability of the zero steady state of Eq. (1.1) according to the linearized system. But the dynamics on the center manifold is topologically equivalent to that of the system on the whole phase space. Therefore, to determine the stability of the zero steady state of Eq. (1.1), we have to compute the normal forms on the center manifold. In the rest of the paper, we focus on the case of* $d=a>b^2/2$, $b<0$, *and* $-b/a<\tau<\tau_0$.

III. NORMAL FORMS ON THE CENTER MANIFOLD FOR A SIMPLE ZERO EIGENVALUE

In this section, we refer the reader to Ref. 13 for notation and general results on the theory of retarded functional differential equations (RFDEs). To determine the stability of the zero steady state of Eq. (1.1) with $d=a>b^2/2$, $b<0$, and $-b/a<\tau<\tau_0$, we have to compute the normal forms on the center manifold. The method we use is based on the center manifold reduction and normal form theory due to Faria and Magalháes (see the Appendix).

Letting a, b, and τ be fixed and considering d as a bifurcation parameter, we shall investigate the dynamics of system (1.1) when $d=a$ and $\tau\neq b/a$. In this case, the characteristic equation has a simple zero solution. In the following, we shall use the procedure as shown in section A to compute the normal form of system (1.1) associated with this zero eigenvalue. Here $\Lambda_0=\{0\}$.

For convenience, we introduce a new parameter μ by considering $d=a+\mu$ so that $\mu=0$ is a steady state bifurcation value of Eq. (1.1). Furthermore, rescale the time by $t\mapsto t/\tau$ to normalize the delay (1.1) and then system (1.1) can be written as Eq. (A1) in the phase space $C=C([-1,0],\mathbb{R}^2)$, with $L:C\to\mathbb{R}^2$ given by

$$L(\mu)(\varphi)=\tau\begin{bmatrix}\varphi_2(0)\\ -a\varphi_1(0)-b\varphi_2(0)+(a+\mu)\varphi_1(-1)\end{bmatrix},\tag{3.1}$$

and $F:C\to\mathbb{R}^2$ given by

$$F(\varphi,\mu)=\frac{\tau f''(0)}{2!}\begin{bmatrix}0\\ \varphi_1^2(-1)\end{bmatrix}+\frac{\tau f'''(0)}{3!}\begin{bmatrix}0\\ \varphi_1^3(-1)\end{bmatrix}+O(|\varphi|^4),\tag{3.2}$$

where $\varphi=(\varphi_1,\varphi_2)^T\in C$. For system (1.1), the function of bounded variation $\eta_\mu:[-1,0]\to\mathbb{R}^2\times\mathbb{R}^2$ is defined by

$$\eta_\tau(\theta)=\tau\delta_0 M-\tau\delta_1 N,$$

where $\delta_\nu=\delta(\theta+\nu)$ is the Dirac distribution at the point $\theta=-\nu$, and M and N are defined by Eq. (2.3) with $d=a+\mu$. Moreover, we can choose Φ and Ψ as follows:

$$\Phi(\theta)=\begin{pmatrix}1\\0\end{pmatrix},\quad -1\leqslant\theta\leqslant 0$$

and

$$\Psi(s)=\frac{1}{b+a\tau}(b,1),\quad 0\leqslant s\leqslant 1$$

such that $\langle\Phi,\Psi\rangle=1$, $\dot{\Phi}=\Phi B$ and $\dot{\Psi}=-B\Psi$, where $B=0$. So, it follows from Eq. (A7) that

$$\frac{1}{2}f_2^1(x,y,\mu)=\frac{\tau}{b+a\tau}\left\{\mu x+\mu y_1(-1)+\frac{f''(0)}{2!}[x+y_1(-1)]^2\right\},$$

$$\frac{1}{2}f_2^2(x,y,\mu)=\tau(I-\pi)X_0\left\{\begin{matrix}0\\ \mu[x+y_1(-1)]+\frac{f''(0)}{2!}[x+y_1(-1)]^2\end{matrix}\right\},$$

$$\frac{1}{3!}f_3^1(x,y,\mu)=\frac{\tau}{b+a\tau}\left\{\frac{f'''(0)}{3!}[x+y_1(-1)]^3\right\},$$

$$\frac{1}{3!}f_3^2(x,y,\mu)=(I-\pi)X_0\left\{\begin{matrix}0\\ \frac{\tau f'''(0)}{3!}[x+y_1(-1)]^3\end{matrix}\right\}.$$

Then it follows from Eq. (A14) that

$$\tfrac{1}{2}g_2^1(x,0,\mu)=\mathrm{Proj}_{[\mathrm{Im}(M_2^1)]^c}\tfrac{1}{2}f_2^1(x,0,\mu).\tag{3.3}$$

Since $B=0$ it is easy to check that

$$[\mathrm{Im}(M_2^1)]^c=\mathrm{span}\{x^2,x\mu,\mu^2\}.\tag{3.4}$$

It follows from Eqs. (3.5) and (3.4) that

$$g_2^1(x,0,\mu)=\frac{2\tau}{b+a\tau}\left[\mu x+\frac{f''(0)}{2!}x^2\right],\tag{3.5}$$

and then the normal form of Eq. (1.1) on the invariant local center manifold $y=0$ is given by (up to the second order terms)

$$\dot{x}=\frac{\tau}{b+a\tau}\left[\mu x+\frac{f''(0)}{2!}x^2\right]+\text{h.o.t.}\tag{3.6}$$

If $f''(0)=0$, we have to compute $g_3^1(x,0,\mu)$. From Eq. (A13) and the fact that $\mathrm{Ker}(M_2^1)^c=0$, we have

$$U_2(x,\mu)=M_2^{-1}P_{I,2}\begin{bmatrix}f_2^1(x,0,\mu)\\ f_2^2(x,0,\mu)\end{bmatrix}=\begin{bmatrix}0\\ h_2(\theta)(x,\mu)\end{bmatrix},$$

where $h_2(\theta)=[h_2^{(1)}(\theta)(x,\mu),h_2^{(2)}(\theta)(x,\mu)]^T$ is the unique solution in $U_2^2(Q^1)$ of the equation

$$(M_2^2h_2)(x,\mu)=(I-\pi)X_0\begin{pmatrix}0\\ 2\tau\mu x\end{pmatrix}.\tag{3.7}$$

Writing $h_{2q}=[h_{2q}^{(1)},h_{2q}^{(2)}]^T\in Q^1$, in the general form, we have

$$\begin{aligned}h_2^{(i)}(\theta)(x,\mu)&=\sum_{|q|=2}h_{2q}^{(i)}(\theta)(x,\mu)^q\\&=h_{220}^{(i)}(\theta)x^2+h_{211}^{(i)}(\theta)x\mu+h_{202}^{(i)}(\theta)\mu^2\end{aligned}$$

with $i=1,2$. It follows from Eq. (3.7) that

$$\dot{h}_{220}^{(1)}(\theta)=0,\quad \dot{h}_{211}^{(1)}(\theta)=\frac{2\tau}{b+a\tau},$$

$$\dot{h}_{202}^{(1)}(\theta)=\dot{h}_{220}^{(2)}(\theta)=\dot{h}_{211}^{(2)}(0a)=\dot{h}_{202}^{(2)}(\theta)=0$$

with the boundary conditions

$$ah_{220}^{(1)}(0)-ah_{220}^{(1)}(-1)=0,$$

$$ah_{211}^{(1)}(0)-ah_{211}^{(1)}(-1)=\frac{2\tau}{b+a\tau},$$

$$h_{202}^{(1)}(0)-h_{202}^{(1)}(-1)=0,$$

$$h_{220}^{(2)}(0)=0,\quad h_{211}^{(2)}(0)=\frac{2}{b+a\tau},\quad h_{202}^{(2)}(0)=0.$$

Solving the above equations with the boundary conditions gives

$$h_{220}(\theta)=\begin{pmatrix}c_1\\0\end{pmatrix},\quad h_{211}(\theta)=\begin{pmatrix}\frac{2\tau}{b+a\tau}\theta+c_2\\ \frac{2}{b+a\tau}\end{pmatrix},\quad h_{202}(\theta)=\begin{pmatrix}c_3\\0\end{pmatrix},$$

where c_i, $i=1,2,3$, are constants and can be determined by

$$h_{2q}(\theta)\in Q^1=\{\varphi\in C^1:\langle\Psi,\varphi\rangle=0\}.$$

A simple calculation shows that $c_1=0$, $c_2=a\tau^2-2/(b+a\tau)^2$, $c_3=0$.

So,

$$U_2(x,\mu)=\begin{bmatrix}0\\ \dfrac{2\tau}{b+a\tau}\theta+\dfrac{a\tau^2-2}{(b+a\tau)^2}\\ \dfrac{2}{b+a\tau}\end{bmatrix}x\mu.$$

Using the change of variables of the form $(x,y)=(\hat{x},\hat{y})+1/2\,U_2(\hat{x},\mu)$ in Eq. (A4), we get (after dropping the hats)

$$\frac{1}{3!}\tilde{f}_3^1=-\frac{\tau(2\tau b+a\tau^2+2)}{2(b+a\tau)^3}x\mu^2+\frac{\tau f'''(0)}{6(b+a\tau)}[x+y_1(-1)]^3.$$

Since $B=0$, we have

$$[\mathrm{Im}(M_3^1)]^c=\mathrm{span}\{x^3,x^2\mu,x\mu^2,\mu^3\},\tag{3.8}$$

and then we have

$$\frac{1}{3!}g_3^1(x,0,\mu)=\mathrm{Proj}_{[\mathrm{Im}(M_3^1)]^c}\frac{1}{3!}\tilde{f}_3^1(x,0,\mu)$$
$$=-\frac{\tau(2\tau b+a\tau^2+2)}{2(b+a\tau)^3}x\mu^2+\frac{\tau f'''(0)}{6(b+a\tau)}x^3,$$

which, together with Eq. (3.6), implies that when $f''(0)=0$ the normal form on the invariant local center manifold $y=0$ is given by (up to the third order terms)

$$\dot{x}=\nu x+\frac{\tau f'''(0)}{6(b+a\tau)}x^3+\mathrm{h.o.t.}\tag{3.9}$$

where $\nu=\tau/b+a\tau\mu-\tau(2\tau b+a\tau^2+2)\mu^2/(2(b+a\tau)^3)$.

From Lemma 2.2, the center manifold by Carr[14] and the bifurcation theorem,[13,15] the dynamics of the delay differential equation (1.1) is topologically equivalent to that of Eq. (3.9) at the sufficiently small neighborhood of $\mu=0$. In addition, notice that in a sufficiently small neighborhood of $\mu=0$ the sign of ν is completely determined by that of μ provided that $d=a>b^2/2$, $b<0$, and $-b/a<\tau<\tau_0$. So, by the normal forms on the center manifold Eqs. (3.6) and (3.9) the following two theorems follow immediately.

Theorem 3.1: *Assume that $d=a>b^2/2$, $b<0$, and $-b/a<\tau<\tau_0$.*

(i) *If $f''(0)\neq 0$, then the zero solution of Eq. (1.1) is unstable.*
(ii) *If $f''(0)=0$, then the zero solution of Eq. (1.1) is stable for $f'''(0)<0$ and unstable for $f'''(0)>0$.*

Theorem 3.2: *Assume that $a>b^2/2$, $b<0$, $-b/a<\tau<\tau_0$, and $d=a+\mu$ with $|\mu|$ being a sufficiently small positive number.*

(i) *The zero solution of Eq. (1.1) is asymptotically stable for $\mu<0$ and unstable for $\mu>0$.*
(ii) *When $f''(0)=0$, Eq. (1.1) undergoes a supercritical pitchfork bifurcation at the zero solution.*
(iii) *When $f''(0)\neq 0$, Eq. (1.1) undergoes a transcritical bifurcation at the zero solution.*

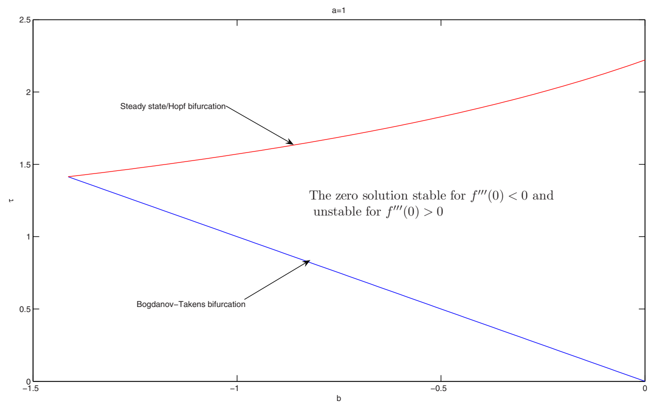

FIG. 1. (Color online) When $f''(0)=0$, $d=a>b^2/2$, and $b<0$, the stability region of the zero solution of Eq. (1.1) in the $b-\tau$ plane is qualitatively equivalent to what is shown in the figure. Here we take $a=1$.

According to Theorem 3.1, if $f''(0)=0$, the stability region for the zero solution of Eq. (1.1) with $d=a>b^2/2$, $b<0$ can be geometrically plotted in Fig. 1.

IV. NUMERICAL SIMULATIONS

In this section, we present numerical simulations of Eq. (1.1) to verify the theoretical results obtained in the previous sections. To this aim, we first define the step as 0.06 in numerical simulations and then the initial condition for the delay differential equation (1.1) is chosen as follows:

$$x(t)=\begin{cases}0, & \text{for } -\tau\leqslant t<-0.06,\\ \dfrac{x_0}{0.06}t-x_0, & \text{for } -0.06<t\leqslant 0,\end{cases}$$
$$\text{and } \dot{x}(t)=y_0,\quad \text{for all } t\in[-\tau,0],\tag{4.1}$$

where x_0, y_0 should be specified in advance in the numerical simulation. In the following, the assignments are split into two parts depending on whether $f''(0)$ is zero or not.

(i) Taking $f(x)=\tanh(x)$, we have $f'(0)=1>0$, $f''(0)=0$, $f'''(0)=-2<0$. For numerical simulations, we further take the parameters $d=a=1$, $b=-0.5$ such that $a>b^2/2$. It follows from Lemma 2.2 that $\tau_0\doteq 1.8285$. So, Theorem 3.1 implies that when $-b/a=0.5<\tau<1.8285=\tau_0$ the zero solution of Eq. (1.1) is stable as illustrated in Figs. 2 and 3. In order to investigate the stability and bifurcation of the zero solution of Eq. (1.1) when d crosses the critical value $d=1$ in ascending order, we should consider a small perturbation of $d=1$. To this aim, we take $f(x)=\tanh(x)-0.1x$ and the result of the numerical simulation is illustrated in Fig. 4, showing the zero solution of Eq. (1.1) is stable. However, when $f(x)=\tanh(x)+0.1x$, the result of the numerical simulation shows that the zero solution of Eq. (1.1) is unstable with two stable nonzero steady states, one being $x\doteq 0.5838$ and the other $x\doteq -0.5838$, emerge, shown in Fig. 5. So, Eq. (1.1) undergoes a pitchfork bifurcation at the zero solution. These numerical results are exactly consistent with the conclusions (i) and (ii) of Theorem 3.2.

(ii) Taking $f(x)=\tanh(x+1)-\tanh(1)$, then

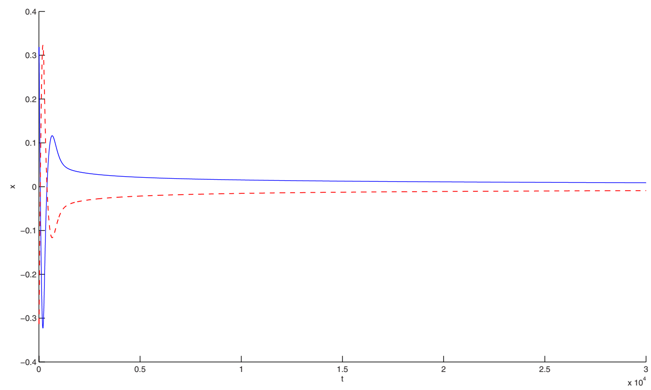

FIG. 2. (Color online) Numerical simulations of Eq. (1.1) with $a=1$, $b=-0.5$, $f(x)=\tanh(x)$, and $\tau=0.6>-b/a=0.5$, showing that the zero solution of Eq. (1.1) is stable. Here the initial condition is given by Eq. (4.1) with $(x_0,y_0)=(0.3,0.1)$ and $(x_0,y_0)=(-0.3,-0.1)$, respectively, for the solid line and the dashed line.

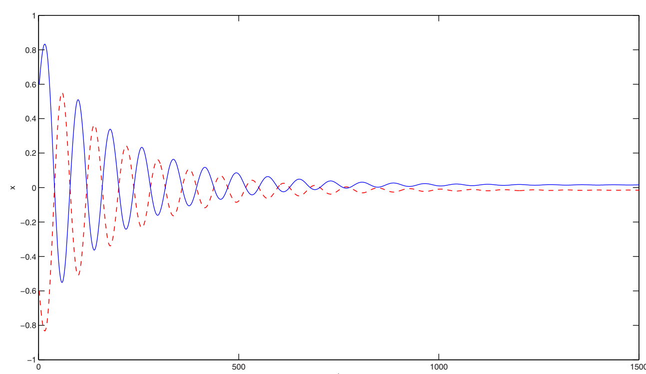

FIG. 3. (Color online) Numerical simulations of Eq. (1.1) with $\tau=1.7<\tau_0 \doteq 1.8285$ and a, b, $f(x)$ as indicated in Fig. 2, showing that the zero solution of Eq. (1.1) is stable. Here the initial condition is given by Eq. (4.1) with $(x_0,y_0)=(0.6,0.5)$ and $(x_0,y_0)=(-0.6,-0.5)$, respectively, for the solid line and the dashed line.

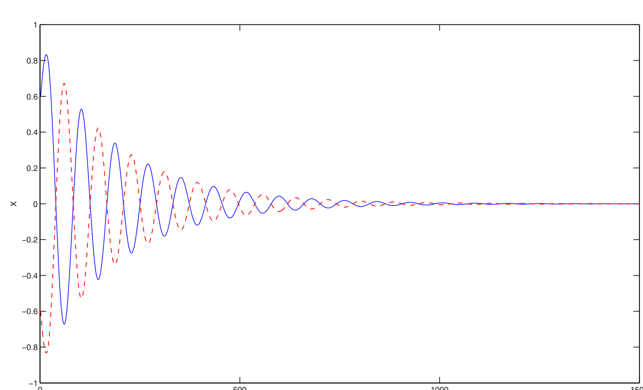

FIG. 4. (Color online) Numerical simulations of Eq. (1.1) with $f(x)=\tanh(x)-0.1x$, $\tau=1.7$ and a, b as indicated in Fig. 2, showing that the zero solution of Eq. (1.1) is stable. Initial conditions are the same as Fig. 3.

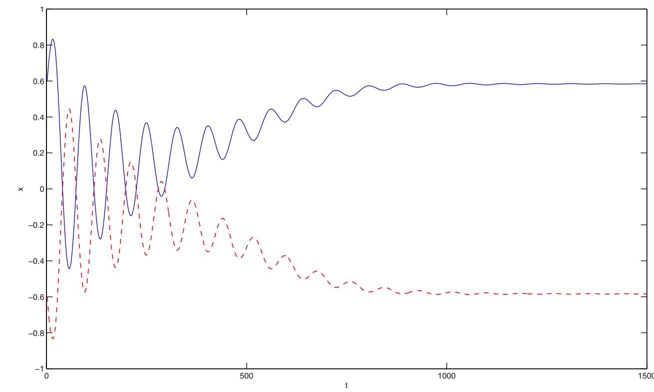

FIG. 5. (Color online) Numerical simulations of Eq. (1.1) with $f(x)=\tanh(x)+0.1x$, $\tau=1.7$ and a, b as indicated in Fig. 2, showing that the zero solution of Eq. (1.1) is unstable and two stable nonzero steady states emerge. Initial conditions are the same as Fig. 3.

$$f'(0)=1-\left(\frac{e^2-1}{e^2+1}\right)^2,$$

$$f''(0)=-2\left(\frac{e^2-1}{e^2+1}\right)\left[1-\left(\frac{e^2-1}{e^2+1}\right)^2\right]<0.$$

In the following numerical simulations, we set $a=1-((e^2-1)/(e^2+1))^2$, $b=-0.3$ and then $a>b^2/2$. From Lemma 2.2, we have $\tau_0 \doteq 2.8575$. Figure 6 illustrates the numerical results of Eq. (1.1) with $\tau=2.4$. Considering a small perturbation of d at the critical value $d=a=1-((e^2-1)/(e^2+1))^2$ by setting $f(x)=\tanh(x+1)-\tanh(1)-0.1x$, then Fig. 7 illustrates numerical simulations of Eq. (1.1) with $\tau=2.4<\tau_0 \doteq 2.8575$ between $-b/a \doteq 0.7143$ and $\tau_0 \doteq 2.8575$. In this case Eq. (1.1) has two steady states, $x=0$ and $x \doteq -0.3555$. The zero solution is stable and the nonzero steady state is unstable. If setting $f(x)=\tanh(x+1)-\tanh(1)+0.1x$ and $\tau=2.4$, then the zero solution of Eq. (1.1) becomes unstable and a nonzero steady state, $x \doteq 0.3555$, becomes stable, as shown in Fig. 8. Thus, Eq. (1.1) undergoes a transcritical bifurcation at the zero solution. This is consistent with conclusion (*iii*) of Theorem 2.

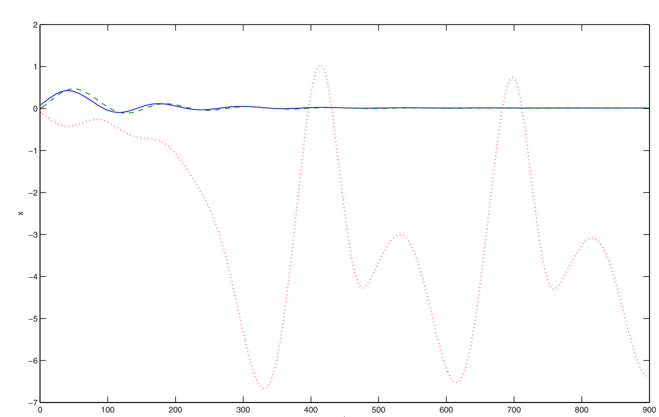

FIG. 6. (Color online) Numerical simulations of Eq. (1.1) with $f(x)=\tanh(x+1)-\tanh(1)$ and a, b, τ as indicated in Fig. 7. Here the initial condition is given by Eq. (4.1) with $(x_0,y_0)=(0.1,0.2)$ for the solid line, $(x_0,y_0)=(0.01,0.2)$ for the dashed line, $(x_0,y_0)=(-0.1,-0.2)$ for the dotted line.

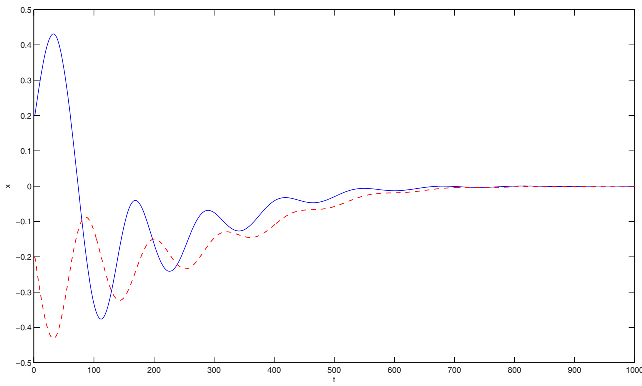

FIG. 7. (Color online) Numerical simulations of Eq. (1.1) with $a=1-e^{2}-1`e^{2}+1$, $b=-0.3$, and $f(x)=\tanh(x+1)-\tanh(1)-0.1x$, $0.7143\doteq -b/a<\tau=2.4<\tau_0\doteq 2.8575$. Here the initial condition is given by Eq. (4.1) with $(x_0,y_0)=(0.2,0.2)$ for the solid line and $(x_0,y_0)=(-0.2,-0.2)$ for the dashed line.

V. CONCLUSIONS

In this paper, we have investigated the stability and multiple bifurcations of a damped harmonic oscillator with delayed feedback near zero eigenvalue singularity. It was shown that there are steady-state bifurcation, Bogdanov–Takens bifurcation, triple zero, and Hopf-zero singularities by analyzing the distribution of the roots of the associated characteristic equation. A center manifold reduction and normal form technology for RFDEs are used to demonstrate the presence of pitchfork and transcritical bifurcations in the damped harmonic oscillator with delayed feedback for the case when the characteristic equation has a single zero root. For this case, the stability of the zero solution is determined by the higher-order derivatives of the restoring force f and the types of bifurcations is completely determined by whether the second-order derivative of the restoring force f is zero or not. The present paper is complementary to the previous work[1,4,10–12] and demonstrate that the damped harmonic oscillator with delayed feedback near zero eigenvalue singularity exhibits the presence of multibifurcations like steady-state bifurcation, Bogdanov–Takens bifurcation, triple zero singularity, Hopf-zero singularity, and even for the simplest case, the simple zero eigenvalue singularity, multistability is possible (see Fig. 5).

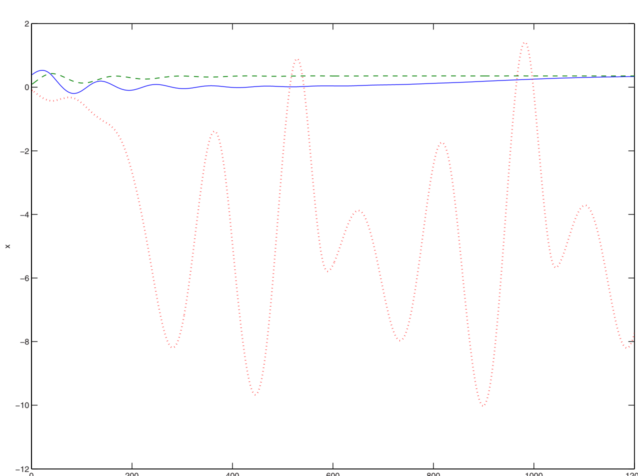

FIG. 8. (Color online) Numerical simulations of Eq. (1.1) with $f(x)=\tanh(x+1)-\tanh(1)+0.1x$ and a, b, τ as indicated in Fig. 7. Here the initial condition is given by Eq. (4.1) with $(x_0,y_0)=(0.4,0.2)$ for the solid line, $(x_0,y_0)=(0.1,0.2)$ for the dashed line, $(x_0,y_0)=(-0.1,-0.2)$ for the dotted line.

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APPENDIX: FARIA AND MAGALHãS NORMAL FORMS

Let $C=C([-1,0],\mathbb{R}^n)$ is the Banach space of continuous functions from $[-1,0]$ into $\mathbb{R}^n$ with supremum norm. We define $z_t\in C$ as $z_t(\theta)=z(t+\theta)$, $-1\geqslant\theta\leqslant 0$. Let us consider the following parameterized family of nonlinear RFDEs with an equilibrium point at the origin:

$$\dot{z}(t)=L(\mu)(z_t)+F(z_t,\mu), \tag{A1}$$

where $\mu\in V$, a neighborhood of zero in $\mathbb{R}^s$, is considered as a parameter, $L:C\times V\to\mathbb{R}^n$ is a parameterized family of bounded linear operators from and $F:C\times V$ is a C^k function $(k\geqslant 2)$, with $F(0,\mu)=0$, $D_1F(0,\mu)=0$ for all $\mu\in\mathbb{R}^s$.

Denote the characteristic equation of the linearized equation at the origin by

$$\det\Delta(\lambda,\mu)=\lambda I-L(e^{\lambda\cdot}I)=0,$$

where I is the $n\times n$ identity matrix. Let $A(\mu)$ be the infinitesimal generator of the flow for the linear system $\dot{z}(t)=L(\mu)z_t$, with spectrum $\sigma[A(\mu)]$. Then we have

$$\sigma[A(\mu)]=\{\lambda\in\mathbb{C}:\det\Delta(\lambda,\mu)=0\}.$$

From the Riesz representation theorem the linear map L_τ can also be expressed in integral form as the following:

$$L(\mu)(\varphi)=\int_{-1}^{0}[d\eta_\mu(\theta)]\varphi(\theta),$$

where $\eta_\mu:[-1,0]\to\mathbb{R}^n\times\mathbb{R}^n$ is a function of bounded variation. Let $\mathbb{R}^{n*}$ be the n-dimensional vector space of row vectors, define $C^*=C([-1,0],\mathbb{R}^{n*})$ and the adjoint bilinear form on $C^*\times C$ as follows:

$$\langle\psi(s),\phi(\theta)\rangle=\psi(0)\phi(0)-\int_{-1}^{0}\int_{0}^{\theta}\psi(\xi-\theta)d\eta_\mu(\theta)\phi(\xi)d\xi, \tag{A2}$$

with $\psi(s)\in C^*$, $\phi(\theta)\in C$. The formal adjoint operator $A^*(\mu)$ of $A(\mu)$ is defined as the infinitesimal generator for the solution operator of the adjoint equation in C^*,

$$\dot{w}(t)=-\int_{-1}^{0}w(t-\theta)[d\eta_\mu(\theta)].$$

Note that the eigenvalues of $\sigma[A(\mu)]$ with zero real parts shall play an important role in the bifurcation theory of

RFDEs. For simplicity, throughout the rest of the paper, we assume that when $\mu=0$ the characteristic equation (A) has roots with zero real parts, i.e., taking $\mu=0$ as the critical value. Let $A_0 \equiv A(0)$ and denote Λ_0 by

$$\Lambda_0 = \{\lambda \in \sigma(A_0) : \mathrm{Re}\,\lambda = 0\} \neq \emptyset.$$

Using the formal adjoint theory for FDEs in Ref. 13, the phase space C can be decomposed by Λ_0 as $C=P\oplus Q$, where P is the generalized eigenspace associated with the eigenvalues in Λ_0, $Q=\{\phi\in C:\langle\psi,\phi\rangle=0$ for all $\psi\in P^*\}$, and the dual space P^* is the generalized eigenspace for $A^*(0)$ associated with the eigenvalues in Λ_0. For dual bases Φ and Ψ of P and P^*, respectively, such that $\langle\Psi(s),\Phi(\theta)\rangle=I_p$, where $p=\dim P$ and I_p is a $p\times p$ identity matrix, there exists a $p\times p$ real matrix B with $\sigma(B)=\Lambda_0$, which satisfies simultaneously

$$\dot{\Phi} = \Phi B \quad \text{and} \quad -\dot{\Psi} = B\Psi.$$

As shown in Refs. 16 and 17, an appropriate phase space for considering normal forms of Eq. (A1) is the Banach space BC of functions from $[-1,0]$ into $\mathbb{R}^n$ which are uniformly continuous on $[-1,0)$ and with a jump discontinuity at 0. The elements of BC have the form $\varphi+X_0\beta$, where $\varphi\in C$, $\beta\in\mathbb{R}^n$, where

$$X_0(\theta) = \begin{cases} I, & \theta = 0, \\ 0, & -r \leqslant \theta < 0, \end{cases}$$

so that BC is identified with $C\times\mathbb{R}^n$ with the norm $|\varphi+X_0\beta|=|\varphi|_C+|\beta|_{\mathbb{R}^n}$.

Let $\pi: BC\to P$ denote the projection

$$\pi(\varphi + X_0\beta) = \Phi[\langle\Psi,\varphi\rangle + \Psi(0)\beta], \quad \varphi \in C, \quad \beta \in \mathbb{R}^n,$$

and then the decomposition $C=P\oplus Q$ yields a decomposition of BC by Λ_0 as the topological direct sum

$$BC = P \oplus \mathrm{Ker}\,\pi$$

with the property $Q \subsetneq \mathrm{Ker}\,\pi$, where Q is an infinite dimensional complementary subspace of P in C as shown above. According to the above decomposition we now decompose $z_t\in C^1$ in Eq. (A1) as $z_t=\Phi x(t)+y$, with $x(t)\in\mathbb{R}^2$ and $y\in Q^1\equiv Q\cap C^1$, where C^1 is the subset of C consisting of continuously differentiable functions. Note that μ is a parameter and is considered as a variable. So, letting $L_0\equiv L(0)$, we rewrite system (A1) as

$$\dot{z}(t) = L_0 z_t + [L(\mu) - L_0]z_t + F(z_t,\mu) \tag{A3}$$

and then system (A1) under the composition $z_t=\Phi x(t)+y$ can be decomposed as a system of abstract ODEs in $\mathbb{R}^p\times\mathrm{Ker}\,\pi$, as

$$\begin{aligned} \dot{x} &= Bx + \Psi(0)\{[L(\mu) - L_0](\Phi x + y) + F(\Phi x + y,\mu)\}, \\ \dot{y} &= A_{Q^1}y + (I-\pi)X_0\{[L(\mu) - L_0](\Phi x + y) + F(\Phi x + y,\mu)\}, \end{aligned} \tag{A4}$$

where

$$A_{Q^1}y = \dot{y} + X_0[L_0(y) - \dot{y}(0)] \tag{A5}$$

is the restriction of A as an operator from Q^1 into $\mathrm{Ker}\,\pi$.

For $u\in C$, $\mu\in V$, considering the formal Taylor expansion,

$$L(\mu)(u) = L_0 u + L_1(\mu)u + \frac{1}{2}L_2(\mu)u + \cdots,$$

$$F(u,\mu) = \frac{1}{2}F_2(u,\mu) + \frac{1}{3!}F_3(u,\mu) + \cdots,$$

the nonlinear term of order j the variables (u,μ) for Eq. (A3) is given by

$$\frac{1}{(j-1)!}L_{j-1}(\mu)u + \frac{1}{j!}F_j(u,\mu).$$

So, system (A4) can be written as

$$\begin{aligned} \dot{x} &= Bx + \sum_{j\geqslant 2}\frac{1}{j!}f_j^1(x,y,\mu), \\ \dot{y} &= A_{Q^1}y + \sum_{j\geqslant 2}\frac{1}{j!}f_j^2(x,y,\mu), \end{aligned} \tag{A6}$$

where

$$\begin{aligned} \frac{1}{j!}f_j^1(x,y,\mu) &= \Psi(0)\left[\frac{1}{(j-1)!}L_{j-1}(\mu)(\Phi x + y) + \frac{1}{j!}F_j(\Phi x + y,\mu)\right], \\ \frac{1}{j!}f_j^2(x,y,\mu) &= (I-\pi)X_0\left[\frac{1}{(j-1)!}L_{j-1}(\mu)(\Phi x + y) + \frac{1}{j!}F_j(\Phi x + y,\mu)\right]. \end{aligned} \tag{A7}$$

As for autonomous ODEs in $\mathbb{R}^n$, the normal forms are obtained by a recursive process of changes of variables. At a step j, the terms of order $j\geqslant 2$ are computed from the terms of the same order and from the terms of lower orders already computed in previous steps. Assume that steps of orders $2,3,\ldots,j-1$ have already been performed, leading to

$$\begin{aligned} \dot{x} &= Bx + \sum_{l=2}^{j-1}\frac{1}{l!}g_l^1(x,y,\mu) + \frac{1}{j!}\tilde{f}_j^1(x,y,\mu) + \text{h.o.t.}, \\ \dot{y} &= A_{Q^1}y + \sum_{l\geqslant 2}^{j-1}\frac{1}{l!}g_l^2(x,y,\mu)\frac{1}{j!}\tilde{f}_j^2(x,y,\mu) + \text{h.o.t.}, \end{aligned} \tag{A8}$$

where we denote by $\tilde{f}_j=(\tilde{f}_j^1,\tilde{f}_j^2)$ the terms of order j in (x,y,μ) obtained after the previous transformations of variables and h.o.t. stands for higher order terms. Following the algorithm of Refs. 16 and 17 at step j, using a change of variables of the form

$$(x,y) = (\hat{x},\hat{y}) + U_j(\hat{x},\mu) \equiv (\hat{x},\hat{y}) + [U_j^1(\hat{x},\mu), U_j^2(\hat{x},\mu)],$$

where x, $\hat{x}\in\mathbb{R}^p$, y, $\hat{y}\in Q^1$, and $U_j^1:\mathbb{R}^{p+s}\to\mathbb{R}^p$, $U_j^2:\mathbb{R}^{p+s}\to Q^1$ are homogeneous polynomials of degree j in $\hat{x}$, system (A6) can be put into the normal form, after dropping the hats for simplification of notations,

$$\dot{x} = Bx + \sum_{j\geq 2} \frac{1}{j!} g_j^1(x,y,\mu),$$

$$\dot{y} = A_{Q^1} y + \sum_{j\geq 2} \frac{1}{j!} g_j^2(x,y,\mu), \tag{A9}$$

where g_j^1, $g_j^2 (j\geq 2)$ are the terms of order j and given by

$$g_j^1(x,y,\mu) = \tilde{f}_j^1(x,y,\mu) - [D_x U_j^1(x,\mu)Bx - BU_j^1(x)],$$

$$g_j^2(x,y,\mu) = \tilde{f}_j^2(x,y,\mu) - [D_x U_j^2(x,\mu)Bx - A_{Q^1} U_j^2(x)]. \tag{A10}$$

Further, if the nonresonance conditions relative to $\Lambda_0 \subset \sigma(A_0)$,

$$(q,\bar{\lambda}) \neq r, \quad \text{for all } r \in \sigma(A_0) \setminus \Lambda_0, \quad q \in \mathbb{N}_0^p, \ |q| \geq 0 \tag{A11}$$

with $\bar{\lambda} = (\lambda_1, \ldots, \lambda_p)$, where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues in Λ_0, counting multiplicities, and $(q,\bar{\lambda}) = q_1\lambda_1 + \cdots + q_p\lambda_p$, are satisfied, then the locally invariant manifold for system (A1) tangent to P at zero must be $y=0$ and the flow on this manifold is given by the p-dimensional ODE

$$\dot{x} = Bx + \frac{1}{2!} g_2^1(x,0,\mu) + \frac{1}{3!} g_3^1(x,0,\mu) + \text{h.o.t.} \tag{A12}$$

The nonlinear terms in Eq. (A9) are in normal form in the classical sense with respect to the matrix B. In applications, $g_j^1(x,0,\mu)$ usually can be determined by the following procedure.

Define A.1: *For* $j\geq 2$, let M_j *denote the operator defined in* $V_j(\mathbb{R}^{p+s} \times \text{Ker}\,\pi)$, *with values in the same space, by*

$$M_j(h_1,h_2) = (M_j^1 h_1, M_j^2 h_2),$$

$$(M_j^1 h_1)(x,\mu) = D_x h_1(x,\mu)Bx - Bh_1(x,\mu),$$

$$(M_j^2 h_2)(x,\mu) = D_x h_2(x,\mu)Bx - A_{Q_1}[h_2(x,\mu)],$$

with domain $D(M_j) = V_j^{p+s}(\mathbb{R}^{p+s}) \times V_j^{p+s}(Q^1)$. *Here, we use the notation* $V_j^{p+s}(Y)$ *to denote the space of homogeneous polynomials of degree* j *in* $p+s$ *variables* $x = (x_1, x_2, \ldots, x_p, \mu_1, \mu_2, \ldots, \mu_s) \in \mathrm{R}^{p+s}$, *with coefficients in* Y, *for* Y *a Banach space.*

According to Refs. 16 and 17, we have

$$U_j(x) = M_j^{-1} P_{I,j} \tilde{f}_j(x,0,\mu) \in \text{Ker}(M_j)^c \tag{A13}$$

and then

$$g_j^1(x,0,\mu) = (I - P_{I,j})\tilde{f}_j^1(x,0,\mu) \in \text{Im}(M_j^1)^c, \tag{A14}$$

where $P_{I,j} = (P_{I,j}^1, P_{I,j}^2)$ is the projection of $V_j^{p+s}(\mathbb{R}^{p+s}) \times V_j^{p+s}(\text{Ker}\,\pi)$ on $Im(M_j^1) \times Im(M_j^2)$.

Remark A.1: This appendix briefly recollected the normal form technique for delay differential equations due to Faria and Magalhaes.[16,17] *In this method, the normalizing transformations are used to linearize the center manifold, so that the normalized restriction to this manifold can be computed by setting the noncritical variables* (y) *to zero. As mentioned above, such a linearization requires certain* (*generic*) *nonresonance condition (A11). Actually, this condition is redundant for the existence of the center manifold and can be completely avoided, if one first computes the expansion of a center manifold and then normalize on it* (*these steps can be combined*), *referring to Ref. 18 for this technique.*

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